ISRAEL JOURNAL OF MATHEMATICS 100 (1997), 29–59

DEGENERATE PRINCIPAL SERIES AND LOCAL THETA CORRESPONDENCE II

BY

SOO TECK LEE AND CHEN-BO ZHU

Department of Mathematics, National University of Singapore
10 Kent Ridge Crescent, Singapore 119260, Republic of Singapore
e-mail: matleest@leonis.nus.sg matzhucb@leonis.nus.sg

ABSTRACT

Following our previous paper [LZ] which deals with the group $U(n,n)$, we study the structure of certain Howe quotients $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ which are natural $\mathrm{Sp}(2n,\mathbf{R})$ modules arising from the Oscillator representation associated with the dual pair $(O(p,q),\mathrm{Sp}(2n,\mathbf{R}))$, by embedding them into the degenerate principal series representations of $\mathrm{Sp}(2n,\mathbf{R})$ studied in [L2].

1. Introduction

In an earlier paper [LZ], we consider the dual pair $(U(p,q),U(n,n)) \subseteq \mathrm{Sp}(4(p+q)n,\mathbf{R})$ and examine the relationship between Howe quotients which correspond to determinant characters of $U(p,q)$ and irreducible constituents of the degenerate principal series representations of $U(n,n)$ studied in [L1]. In this paper we shall do the same for the dual pair $(H,G) = (O(p,q),\mathrm{Sp}(2n,\mathbf{R})) \subseteq \mathrm{Sp}(2n(p+q),\mathbf{R})$. More precisely we let $V = \mathbf{R}^{p+q}$ and V^n be the direct sum of n copies of V. As usual, we let $\widetilde{\mathrm{Sp}}(2n(p+q),\mathbf{R})$ be the unique non-trivial double cover of $\mathrm{Sp}(2n(p+q),\mathbf{R})$ and $\mathcal{S}(V^n)$ the Schwartz space of V^n. Then $\widetilde{\mathrm{Sp}}(2n(p+q,\mathbf{R})$ acts on $\mathcal{S}(V^n)$ via the Oscillator representation ω. Let $S \subseteq \mathcal{S}(V^n)$ be the space of Schwartz functions which correspond to polynomials in the Fock model of ω. Let $\Omega^{p,q}$ be the maximal quotient of S which is

Received November 2, 1995

trivial as an $(\mathfrak{o}(p,q), L)$ module, where $L \cong O(p) \times O(q)$ is a maximal compact subgroup of $O(p,q)$. We also let $\Omega^{p,q}(1)$ be the maximal quotient of S on which $(\mathfrak{o}(p,q), L)$ acts by the representation which corresponds to the determinant character $g \longrightarrow \det g$ of $O(p,q)$. It turns out that under certain conditions, these Howe quotients $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ can be embedded into the following degenerate series $\{I^{\pm}(\sigma)\colon\ \sigma \in \mathbf{C}\}$ of $\mathrm{Sp}(2n,\mathbf{R})$ studied in [L2]. Let $P = MN$ be the maximal parabolic subgroup of $\mathrm{Sp}(2n,\mathbf{R})$ where

$$\begin{aligned} M &= \left\{ m_a = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} : a \in \mathrm{GL}(n,\mathbf{R}) \right\}, \\ N &= \left\{ n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in M(n,\mathbf{R}),\ b = b^t \right\}. \end{aligned}$$

For each $\sigma \in \mathbf{C}$, we let $\chi_\sigma^{\pm}\colon P \longrightarrow \mathbf{C}^{\times}$ be the characters given by

$$\chi_\sigma^{\pm}(m_a n_b) = \begin{cases} (\det a)^{\sigma} & \text{if } \det a > 0, \\ \pm|\det a|^{\sigma} & \text{if } \det a < 0, \end{cases}$$

and let $I^{\pm}(\sigma)$ be the corresponding induced representations. The representation spaces for $I^{\pm}(\sigma)$ are respectively

$$\{f \in C^{\infty}(\mathrm{Sp}(2n,\mathbf{R}))\colon f(pg) = \delta(p)^{\frac{1}{2}}\chi_\sigma^{\pm}(p) f(g),\ g \in \mathrm{Sp}(2n,\mathbf{R}), p \in P\},$$

where δ is the modular function for P, and on which $\mathrm{Sp}(2n,\mathbf{R})$ acts by right translation.

COROLLARY 3.2: *Let $\rho_n = \frac{n+1}{2}$. We have G-equivariant embeddings*

$$\varphi\colon\ \Omega^{p,q} \hookrightarrow \begin{cases} I^{+}(\frac{p+q}{2} - \rho_n), & \text{if } p-q \equiv 0 \pmod 4, \\ I^{-}(\frac{p+q}{2} - \rho_n), & \text{if } p-q \equiv 2 \pmod 4, \end{cases}$$

if $p+q$ is even, and

$$\psi\colon\ \Omega^{p,q}(1) \hookrightarrow \begin{cases} I^{-}(\frac{1}{2}), & \text{if } p-q \equiv 0 \pmod 4,\ p+q = n, \\ I^{+}(\frac{1}{2}), & \text{if } p-q \equiv 2 \pmod 4,\ p+q = n, \end{cases}$$

if n is even.

The embedding φ is first given by Kudla and Rallis in [KR1]. In the same paper, they also prove that $I^{\pm}(\sigma)$ are irreducible if and only if $\sigma \notin \rho_n + \mathbf{Z}$.

Since all of $\Omega^{p,q}$, $\Omega^{p,q}(1)$ and $I^{\pm}(\sigma)$ are K-multiplicity free, where $K \cong U(n)$ is a maximal compact subgroup of G, we can easily identify the images of $\Omega^{p,q}$

and $\Omega^{p,q}(1)$ in $I^{\pm}(\sigma)$. In particular we shall show that some of the irreducible unitary submodules of $I^{\pm}(\sigma)$ can be identified with certain Howe quotients $\Omega^{p,q}$ or $\Omega^{p,q}(1)$. Since the module structure of $I^{\pm}(\sigma)$ has been completely determined in [L2], we can deduce the module structure of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$. In particular we can describe a composition series of these $\Omega^{p,q}$ when it is reducible, and determine how the different $\Omega^{p,q}$'s with $p+q$ fixed intersect each other in $I^{\pm}(\sigma)$. In this paper we only consider Howe quotients which are embedded into $I^{+}(\sigma)$. Those Howe quotients which are embedded into $I^{-}(\sigma)$ can be studied in exactly the same way, so we omit them. Here we highlight some of the results:

THEOREM: *Let m be an even positive integer and $\sigma = \frac{m}{2} - \rho_n$.*

(i) *If $2 \leq m \leq n$, then the set $\{\varphi(\Omega^{p,q})\colon p+q = m, p-q \equiv 0 \pmod 4\}$ exhausts all the irreducible unitary submodules of $I^{+}(\sigma)$.*

(ii) *Assume that n is odd and $m = n+1$, so that $\sigma = 0$.*

(a) *If $n \equiv 1 \pmod 4$, then*

$$I^{+}(0) = \bigoplus \{\varphi(\Omega^{p,n+1-p}), 1 \leq p \leq n, p\, odd\}$$

is the direct sum of $\frac{n+1}{2}$ irreducible unitary submodules.

(b) *If $n \equiv 3 \pmod 4$, then*

$$I^{+}(0) = \bigoplus \{\varphi(\Omega^{p,n+1-p}), 0 \leq p \leq n+1, p\, even\}$$

is the direct sum of $\frac{n+3}{2}$ irreducible unitary submodules.

We remark that results in part (ii) are first given in [K]. This is related to an earlier work of Kashiwara and Vergne [KV2] (see also [S]). Recall that if M is a module for a group or an algebra, then the socle of M is the sum of all irreducible submodules of M, and is written Soc (M) (see [GW]). The socle series of M is the ascending chain

$$\mathrm{Soc}^{\,0}(M) \subseteq \mathrm{Soc}^{\,1}(M) \subseteq \mathrm{Soc}^{\,2}(M) \subseteq \cdots$$

of submodules of M defined inductively by setting $\mathrm{Soc}^{\,0}(M) = 0$ and

$$\mathrm{Soc}^{\,r+1}(M)/\mathrm{Soc}^{\,r}(M) = \mathrm{Soc}\,(M/\mathrm{Soc}^{\,r}(M))$$

for any nonnegative integer r. For $m \geq n+2$ (so that $\sigma > 0$), it can be shown that each irreducible constituent in $\mathrm{Soc}^{\,1}(I^{+}(\sigma))$ is either a Howe quotient or the intersection of two Howe quotients; and for $j \geq 2$, each irreducible constituent in $\mathrm{Soc}^{\,j}(I^+(\sigma))$ is either the quotient of one Howe quotient by a submodule in $\mathrm{Soc}^{\,j-1}(I^+(\sigma))$, or the quotient of the intersection of two Howe quotients by a submodule in $\mathrm{Soc}^{\,j-1}(I^+(\sigma))$. More precise statements similar to Proposition 5.10 of [LZ] can be made using detailed results for the various cases given in section 4. We shall leave them to the readers. Thus the reducibility of $I^+(\sigma)$ is fully accounted for by these Howe quotients. This gives the archimedean analog of the results of Kudla and Rallis on ramified degenerate principal series representations of symplectic groups over p-adic field ([KR2]).

On the other hand, the local Howe duality principle [H1] tells us that the representation $\Omega^{p,q}$ always has a unique irreducible quotient $Q^{p,q}$. Notice that $Q^{p,q}$ is the representation $\theta(1)$ associated to the trivial representation of $H = O(p,q)$ via the local theta correspondence. By identifying $Q^{p,q}$ with an irreducible constituent of $I^+(\sigma)$, we obtain the following.

THEOREM: *Suppose that p and q are nonnegative integers such that $p+q$ is even and $p-q \equiv 0 \pmod 4$. Then $Q^{p,q}$ is unitary if and only if either $pq = 0$ or both $p, q \leq n+1$.*

The fact that $Q^{p,q}$ is unitary for $p+q \leq n$ also follows from the more general results of Li on stable dual pairs [Li].

When n is even, our results also reveal a very interesting phenomenon in $I^+(\frac{1}{2})$ that certain Howe quotients corresponding to the determinant character embed as irreducible submodules of certain Howe quotients which correspond to the trivial representation.

THEOREM: *Assume n is even. Then every irreducible constituent of $I^+(\frac{1}{2})$ is unitary. The socle length of $I^+(\frac{1}{2})$ is equal to two, and the set*

$$\{\psi\,(\Omega^{p,q}(1)) : p+q = n, p-q \equiv 2 \,(\mathrm{mod}\ 4)\}$$

exhausts all the irreducible unitary submodules of $I^+(\frac{1}{2})$.

(a) *If $n \equiv 0 \pmod 4$ and p and q are odd integers such that $p+q = n+2$, then we also have embeddings $\varphi\colon \Omega^{p,q} \hookrightarrow I^+(\frac{1}{2})$. Moreover, we have:*
   - (i) *$\Omega^{n-1,1}(1)$ is isomorphic to the unique irreducible submodule of $\Omega^{n+1,1}$.*
   - (ii) *If p is an odd integer such that $3 \leq p \leq n-3$, then $\Omega^{p,n-p}(1)$ and $\Omega^{p-2,n-p+2}(1)$ are isomorphic to the two irreducible submodules of*

$\Omega^{p,n+2-p}$, *respectively.*

(iii) $\Omega^{1,n-1}(1)$ *is isomorphic to the unique irreducible submodule of* $\Omega^{1,n+1}$.

(b) *If* $n \equiv 2 \pmod 4$ *and* p *and* q *are even integers such that* $p+q=n+2$, *then we also have embeddings* $\varphi\colon \Omega^{p,q} \hookrightarrow I^+(\frac{1}{2})$. *Moreover we have:*

(i) $\Omega^{n,0}(1)$ *is isomorphic to* $\Omega^{n+2,0}$.

(ii) *If* p *is an even integer such that* $2 \le p \le n-2$, *then* $\Omega^{p,n-p}(1)$ *and* $\Omega^{p-2,n-p+2}(1)$ *are isomorphic to the two irreducible submodules of* $\Omega^{p,n+2-p}$, *respectively.*

(iii) $\Omega^{0,n}(1)$ *is isomorphic to* $\Omega^{0,n+2}$.

When $I^+(\sigma)$ is reducible, the sizes of all irreducible constituents, as measured by their Gelfand–Kirillov dimension, are also determined. Just as in the case of $U(n,n)$ [LZ], the size of each constituent can be read off from its position in the module diagram of $I^+(\sigma)$. But a new phenomenon emerges for $\mathrm{Sp}(2n,\mathbf{R})$ in that there can be two rows in the module diagram which has the maximal G-K dimension $\frac{n(n+1)}{2}$. For example this is the case in $I^+(\frac{1}{2})$ for n even. The detailed result is given in section 5.

The study of Howe quotients was pioneered by Kudla and Rallis ([KR1] in the archimedean case and [KR2] in the p-adic case), motivated by their work on the extension of the Siegel–Weil formula. In the p-adic case there are only two Howe quotients corresponding to two quadratic spaces (with different Hasse invariants) of a given even dimension and given a discriminant. These two Howe quotients are embedded into the same ramified degenerate series of $\mathrm{Sp}(2n,\mathbf{R})$, and their relationships are completely understood [KR2]. In the real case that we are considering, some results have already been obtained by Kudla and Rallis in [KR1]. For example, they have determined which of the $\Omega^{p,q}$'s are irreducible and which of the $Q^{p,q}$'s are finite dimensional. In fact for $n=2$, complete structural results have been stated and used in their work with Soudry [KRS]. One of our main purpose in this paper is to describe these results for any n and in any range of p,q as explicitly as possible. We hope they will prove to be useful in the further study of Eisenstein series and of L-functions for symplectic groups.

ACKNOWLEDGEMENT: The authors would like to thank the referee for pointing out an inconsistency between the K-types of $\Omega^{p,q}$ and $\mathcal{S}^*(V^n)^H$ in an earlier version (see section 2).

2. $\widetilde{K}$-types of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$

Consider the reductive dual pair

$$(H, G) = (O(p,q), \mathrm{Sp}(2n, \mathbf{R})) \subseteq \mathrm{Sp}(2n(p+q), \mathbf{R}).$$

For any subgroup E of $\mathrm{Sp}(2n(p+q), \mathbf{R})$, $\widetilde{E}$ shall denote the preimage of E under the canonical projection $\widetilde{\mathrm{Sp}}(2n(p+q), \mathbf{R}) \longrightarrow \mathrm{Sp}(2n(p+q), \mathbf{R})$. Let $V = \mathbf{R}^{p+q}$ and let V^n be the direct sum of n copies of V. Then $\widetilde{\mathrm{Sp}}(2n(p+q), \mathbf{R})$ acts on the space $\mathcal{S}(V^n)$ of Schwartz functions on V^n via the Oscillator representation ω. By twisting ω with a character, H will act linearly on $\mathcal{S}(V^n)$:

$$\omega(h) \cdot f(v_1, \dots, v_n) = f(h^{-1} \cdot v_1, \dots, h^{-1} \cdot v_n), \quad (h \in H,\ (v_1, \dots, v_n) \in V^n).$$

We shall assume that this has been done from now on.

Let $S \subseteq \mathcal{S}(V^n)$ be the space of Schwartz functions which correspond to polynomials in the Fock model. Then S is naturally a $(\mathfrak{o}(p,q), O(p) \times O(q)) \times (\mathfrak{sp}(2n, \mathbf{R}), \widetilde{U}(n))$ module. Recall that $\Omega^{p,q}$ is the maximal quotient of S which is trivial as an $(\mathfrak{o}(p,q), O(p) \times O(q))$ module, and $\Omega^{p,q}(1)$ is the maximal quotient of S on which $(\mathfrak{o}(p,q), O(p) \times O(q))$ acts by the representation which corresponds to the determinant character $h \longrightarrow \det h$ of $O(p,q)$. The purpose of this section is to decompose $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ into direct sums of $\widetilde{K}$-types. As in the $U(n,n)$ case [LZ], the $\widetilde{K}$-types in $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ are precisely the contragradient representations of those which occur in certain spaces of tempered distributions. We omit all the proofs in this section because they follow the same line of arguments used in the proofs for the corresponding results in the $U(n,n)$ case, which are given in sections 2 and 3 of [LZ].

We now fix some notations. Recall that $K = U(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbf{R})$. Denote by Λ_n^+ the set of all dominant integral weights for $U(n)$ with respect to the Borel subalgebra $\mathfrak{b}_n^+$ of upper triangular matrices. Λ_n^+ can be identified in the usual way with the set of all n-tuples of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For each $\lambda \in \Lambda_n^+$, ρ^λ shall denote a copy of the irreducible representation of $U(n)$ with highest weight λ. For convenience, we also let

$$\mathbf{1}_n = (1, \dots, 1).$$

Note that $\mathbf{1}_n$ is the highest weight of the determinant character of $U(n)$. In a similar way, every irreducible representation of $\widetilde{K} = \widetilde{U}(n)$ is of the form ρ^λ, but the components of λ can be half integers.

Now ω induces an action of $\widetilde{\mathrm{Sp}}(2n(p+q),\mathbf{R})$ on the space $\mathcal{S}^*(V^n)$ of tempered distributions on V^n in the usual way. Let $\mathcal{S}^*(V^n)^H$ be the space of H-invariant tempered distributions on V^n. Kudla and Rallis showed in [KR1] that it is generated as a $\widetilde{G}$ module by the Dirac distribution. The second-named author later generalized their result to any classical group H of hermitian or skew-hermitian type acting on the direct sum of n-copies of the standard module V, using a different and more direct approach ([Zhu]).

THEOREM 2.1 ([KR1], see also [Zhu]):

(a) *$\mathcal{S}^*(V^n)^H$ is the closed span of the set $\{\omega(g)\delta | g \in \widetilde{G}\}$. Here δ is the Dirac distribution at the origin of V^n.*

(b) *The multiplicity of τ in $\mathcal{S}^*(V^n)^H$ is at most one for any $\tau \in \widehat{\widetilde{K}}$. It is equal to one if and only if the contragradient representation of τ is isomorphic to ρ^λ, $\tau^* \cong \rho^\lambda$, where*

$$\lambda = \frac{p-q}{2}\mathbf{1}_n + (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0, -\beta_s, \ldots, -\beta_1),$$

and α_i, β_i are all even integers satisfying

$$\alpha_1 \geq \cdots \geq \alpha_t \geq 0, \quad \beta_1 \geq \cdots \geq \beta_s \geq 0, \quad t \leq \min(p, n), s \leq \min(q, n).$$

Moreover the projection of δ to such a $\widetilde{K}$-type is not zero.

Remark: Theorem I of [Zhu] states that the $\widetilde{K}$-types in $\mathcal{S}^*(V^n)^H$ are of the form $\tau \cong \rho^\lambda$, where λ is given above. This is incorrect. A correct proof can be obtained by making only a minor change in the original proof.

Next we let $(\Omega^*)^{p,q}(1)$ be the space of tempered distributions consisting of those $\Phi \in \mathcal{S}^*(V^n)$ such that

$$h \cdot \Phi = (\det h)\Phi, \qquad h \in H.$$

Let $X = (x_{ij})_{1\leq i \leq p+q, 1 \leq j \leq n}$ be the natural coordinates of $V^n \cong M_{p+q,n}(\mathbf{R})$. For $1 \leq t \leq \min(p+q, n)$, let

$$\partial_t = \partial_t(X) = \det \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{12}} & \cdots & \frac{\partial}{\partial x_{1t}} \\ \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{22}} & \cdots & \frac{\partial}{\partial x_{2t}} \\ \cdots & \cdots & & \cdots \\ \frac{\partial}{\partial x_{t1}} & \frac{\partial}{\partial x_{t2}} & \cdots & \frac{\partial}{\partial x_{tt}} \end{pmatrix}.$$

Assume that $p + q \leq n$. Then one can verify that for $h \in H$, we have

$$h \cdot (\partial_{p+q}\delta) = (\det h)(\partial_{p+q}\delta).$$

Consequently the tempered distribution $\mathcal{D}$ given by

$$\mathcal{D} = \partial_{p+q}\delta$$

is in the space $(\Omega^*)^{p,q}(1)$. We have the following description of $(\Omega^*)^{p,q}(1)$.

THEOREM 2.2: *$(\Omega^*)^{p,q}(1)$ is nontrivial if and only if $p+q \le n$. If $p+q \le n$, then*

(a) *$(\Omega^*)^{p,q}(1)$ is the closed span of the set $\{\omega(g)\mathcal{D} | g \in \widetilde{G}\}$.*

(b) *The multiplicity of τ in $\Omega^*(1)$ is at most one for any $\tau \in \widehat{\widetilde{K}}$. It is equal to one if and only if the contragradient module of τ is isomorphic to ρ^λ, $\tau^* \cong \rho^\lambda$, where*

$$\lambda = \frac{p-q}{2}\mathbf{1}_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\beta_q, \ldots, -\beta_1),$$

and α_i, β_i are all odd integers satisfying

$$\alpha_1 \ge \cdots \ge \alpha_p \ge 1, \quad \beta_1 \ge \cdots \ge \beta_q \ge 1.$$

Moreover the projection of $\mathcal{D}$ to such a $\widetilde{K}$-type is not zero.

We remark that Przebinda has proved that $(\Omega^*)^{p,q}(1)$ is trivial if $p+q > n$ ([P]).

Notice that there is a non-degenerate $\widetilde{G}$-equivalent pairing between $\Omega^{p,q}$ (resp. $\Omega^{p,q}(1)$) and $\mathcal{S}^*(V^n)^H$ (resp. $(\Omega^*)^{p,q}(1)$), so that their $\widetilde{K}$-types are contragradient to each other.

PROPOSITION 2.3 ([KR1]): *The $\widetilde{K}$-types in $\Omega^{p,q}$ are those representations of the form $\tau \cong \rho^\lambda$, where*

$$\lambda = \frac{p-q}{2}\mathbf{1}_n + (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0, -\beta_s, \ldots, -\beta_1),$$

and α_i, β_i are even integers such that

$$\alpha_1 \ge \cdots \ge \alpha_t \ge 0, \quad \beta_1 \ge \cdots \ge \beta_s \ge 0, \quad t \le \min(p,n), s \le \min(q,n).$$

Moreover each such $\widetilde{K}$-type occurs with multiplicity one.

PROPOSITION 2.4: *$\Omega^{p,q}(1)$ is nonempty if and only if $p+q \le n$. If $p+q \le n$, then the $\widetilde{K}$-types in $\Omega^{p,q}(1)$ are those representations of the form $\tau \cong \rho^\lambda$, where*

$$\lambda = \frac{p-q}{2}\mathbf{1}_n + (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\beta_q, \ldots, -\beta_1),$$

and α_i, β_i are odd integers such that

$$\alpha_1 \geq \cdots \geq \alpha_p \geq 1, \quad \beta_1 \geq \cdots \geq \beta_q \geq 1.$$

Moreover each such $\widetilde{K}$-type occurs with multiplicity one.

The proofs for Theorems 2.1, 2.2 and Propositions 2.3, 2.4 are similar to that of Theorems 3.1, 3.4, Propositions 2.1 and 2.2 and Corollary 3.11 of [LZ]. The strategy is as follows: First we show that the $\widetilde{K}$-types of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ must be of the forms given in Propositions 2.3 and 2.4 and with multiplicity at most one. Then we show that the contragradient representations of those $\widetilde{K}$-types indeed occur in $\mathcal{S}^*(V^n)^H$ and $(\Omega^*)^{p,q}(1)$ by showing that the Dirac distribution or $\mathcal{D}$ has nonzero projections there.

3. Embedding of Howe quotients into degenerate principal series

It is shown in [KR1] that the Howe quotient $\Omega^{p,q}$ can be embedded into a certain degenerate principal series representation of $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$. We shall show that for $p+q = n$, $\Omega^{p,q}(1)$ can be embedded in a similar way. Further if $p+q$ is even, $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ can in fact be embedded into the degenerate series $\{I^{\pm}(\sigma)\colon \sigma \in \mathbf{C}\}$ of $\mathrm{Sp}(2n, \mathbf{R})$ studied in [L2]. Thus we can use the results in [L2] to deduce the structure of these Howe's quotients by identifying their images in the degenerate series.

We shall identify $\widetilde{\mathrm{Sp}} = \widetilde{\mathrm{Sp}}(2(p+q)n, \mathbf{R})$ as a set with

$$\{(g, \varepsilon)\colon g \in \mathrm{Sp}(2(p+q)n, \mathbf{R}), \quad \varepsilon = \pm 1\}.$$

We recall that $G = \mathrm{Sp}(2n, \mathbf{R})$, and $\widetilde{G} = \widetilde{\mathrm{Sp}}(2n, \mathbf{R})$ is the preimage of G in $\widetilde{\mathrm{Sp}}$. We also recall the parabolic subgroup $P = MN$ of $\mathrm{Sp}(2n, \mathbf{R})$. Then $\widetilde{P} = \widetilde{M}\widetilde{N}$ where

$$\widetilde{M} = \{(m_a, \varepsilon)\colon \ m_a \in M, \ \varepsilon = \pm 1\}, \quad \widetilde{N} = \{(n_b, 1)\colon \ n_b \in N\}.$$

Let $\widetilde{\chi}_1$ be the following character of $\widetilde{M}$ of order 4:

$$\widetilde{\chi}_1(m_a, \varepsilon) = \varepsilon \begin{cases} 1 & \text{if } \det a > 0, \\ i & \text{if } \det a < 0. \end{cases}$$

For $\alpha \in \mathbf{Z}/4\mathbf{Z}$, $\sigma \in \mathbf{C}$, let

$$\widetilde{\chi}_\alpha(\sigma)(m_a, \varepsilon) = |\det a|^\sigma \widetilde{\chi}_1(m_a, \varepsilon)^\alpha$$

and extend trivially on $\widetilde{N}$. Let $\widetilde{I}_\alpha(\sigma) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}} \widetilde{\chi}_\alpha(\sigma)$ be the corresponding induced representation of $\widetilde{G}$. The representation space for $\widetilde{I}_\alpha(\sigma)$ is

$$\{f \in C^\infty(\widetilde{G}) \colon f(\widetilde{p}\widetilde{g}) = \delta_{\widetilde{P}}(\widetilde{p})^{\frac{1}{2}} \chi_\alpha(\sigma)(\widetilde{p}) f(\widetilde{g}),\ \forall \widetilde{g} \in \widetilde{G}, \widetilde{p} \in \widetilde{P}\}$$

and on which $\widetilde{G}$ acts by right translation. Here $\delta_{\widetilde{P}}$ is the modular function of the group $\widetilde{P}$ and is given by $\delta_{\widetilde{P}}(\widetilde{p}) = |\det a|^{2\rho_n}$ for $\widetilde{p} = (m_a, \epsilon)(n_b, 1)$ and $\rho_n = \frac{n+1}{2}$.

Now recall the reductive dual pair $(H, G) = (O(p,q), \mathrm{Sp}(2n, \mathbf{R})) \subseteq \mathrm{Sp}(2(p+q)n, \mathbf{R})$, and the action of $\widetilde{\mathrm{Sp}}(2(p+q)n, \mathbf{R})$ on $\mathcal{S}(V^n)$ via the Oscillator representation ω. We can identify P with the subgroup of $\mathrm{Sp}(2n, \mathbf{R})$ fixing $V^n \cong M_{p+q,n}(\mathbf{R})$. Then we have

$$[\omega((m_a, \varepsilon))f](x) = \widetilde{\chi}_\alpha(\beta)((m_a, \varepsilon)) f(xa),$$
$$[\omega((n_b, 1))f](x) = e^{\frac{1}{2} i tr(I_{p,q} x b x^t)} f(x),$$

for $(m_a, \varepsilon) \in \widetilde{M}$, $(n_b, 1) \in \widetilde{N}$, $f \in \mathcal{S}(M_{p+q,n}(\mathbf{R}))$, $x \in M_{p+q,n}(\mathbf{R})$. Here $\beta = \frac{p+q}{2}$, and $\alpha \equiv p - q \pmod 4$. It follows that

$$\omega(\widetilde{p}) \cdot \delta = \widetilde{\chi}_\alpha(\beta)^{-1}(\widetilde{p})\delta, \quad \widetilde{p} \in \widetilde{P}.$$

If we assume further that $p + q = n$, then direct computations show that for $\widetilde{p} = (m_a, \varepsilon)(n_b, 1)$,

$$\omega(\widetilde{p}) \cdot \partial_{p+q}\delta = \widetilde{\chi}_\alpha(\beta)^{-1}(\widetilde{p})(\det a)^{-1} \partial_{p+q}\delta = \widetilde{\chi}_{\alpha+2}(\beta+1)^{-1}(\widetilde{p}) \partial_{p+q}\delta.$$

Let $\mathcal{D}$ be $\delta \in \mathcal{S}^*(V^n)^H$ or $\partial_{p+q}\delta \in (\Omega^*)^{p,q}(1)$ for $p+q = n$. For any $f \in \mathcal{S}(V^n)$, we consider the function

$$\widetilde{\mathcal{D}}_f(\widetilde{g}) = \mathcal{D}(\omega(\widetilde{g})f), \qquad \widetilde{g} \in \widetilde{G}.$$

Then since $\mathcal{D}(\omega(\widetilde{p}\widetilde{g})f) = (\omega(\widetilde{p})^{-1}\mathcal{D})(\omega(\widetilde{g})f)$, the transformation properties of $\mathcal{D}$ under $\widetilde{P}$ implies that $\widetilde{\mathcal{D}}_f$ is in the space of the induced representation $\widetilde{I}_\alpha(\sigma)$, where

$$(1) \quad \begin{array}{lll} \sigma = \frac{p+q}{2} - \rho_n, & \alpha \equiv p - q \pmod 4 & \text{if } \mathcal{D} = \delta, \\ \sigma = \frac{p+q}{2} + 1 - \rho_n = \frac{1}{2}, & \alpha \equiv p - q + 2 \pmod 4 & \text{if } \mathcal{D} = \partial_{p+q}\delta,\ p+q = n. \end{array}$$

Consequently we obtain a map

$$\begin{array}{rcl} \mathcal{S}(V^n) & \longrightarrow & \widetilde{I}_\alpha(\sigma) \\ f & \longrightarrow & \widetilde{\mathcal{D}}_f. \end{array}$$

We now restrict the map to the space $S \subseteq \mathcal{S}(V^n)$. Because of the way Howe's quotients $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ are defined and the transformation property of $\mathcal{D}$, this restriction map factors through $\Omega^{p,q}$ and $\Omega^{p,q}(1)$, respectively. We shall use φ (respectively ψ) to denote this map from $\Omega^{p,q}$ (respectively $\Omega^{p,q}(1)$) to $\widetilde{I}_\alpha(\sigma)$.

THEOREM 3.1: *The maps*

$$\varphi\colon \Omega^{p,q} \to \widetilde{I}_\alpha(\sigma)$$
$$\psi\colon \Omega^{p,q}(1) \to \widetilde{I}_\alpha(\tfrac{1}{2}), \qquad p+q=n,$$

are $\widetilde{G}$-equivariant embeddings. Here σ and α are as in equation (1).

The embedding φ is first given by Kudla and Rallis in [KR1]. For the embedding ψ, the proof is similar to that of Theorem 4.5 of [LZ].

If $p+q$ is even, then $\omega|_{\widetilde{G}}$ factors through G. This induces an action of G on $\mathcal{S}(V^n)$. We shall abuse notations and denote this action of G also by ω. For each $f \in \mathcal{S}(V^n)$, we let $\mathcal{D}_f$ be the function on G given by

$$\mathcal{D}_f(g) = \mathcal{D}(\omega(g)f), \quad g \in G.$$

Let $\chi^+(\sigma)$, $\chi^-(\sigma)$ be the characters of P given by $\chi^+(\sigma) = \widetilde{\chi}_0(\sigma)|_P$, and $\chi^-(\sigma) = \widetilde{\chi}_2(\sigma)|_P$. Thus we have

$$\chi^\pm(\sigma)(p) = \begin{cases} (\det a)^\sigma, & \text{if } \det a > 0, \\ \pm|\det a|^\sigma, & \text{if } \det a < 0, \end{cases}$$

for $p = m_a n_b \in P$. Further let $I^\pm(\sigma) = \operatorname{Ind}_P^G \chi^\pm(\sigma)$ be the corresponding (normalized) induced representation of G. Then we have

COROLLARY 3.2: *Suppose that $p+q$ is even. Then the map*

$$f \longrightarrow \mathcal{D}_f$$

induces the following G-equivariant embeddings:

$$\varphi\colon \Omega^{p,q} \hookrightarrow \begin{cases} I^+(\frac{p+q}{2} - \rho_n), & \text{if } p-q \equiv 0 \pmod 4, \\ I^-(\frac{p+q}{2} - \rho_n), & \text{if } p-q \equiv 2 \pmod 4, \end{cases}$$
$$\psi\colon \Omega^{p,q}(1) \hookrightarrow \begin{cases} I^-(\frac{1}{2}), & \text{if } p-q \equiv 0 \pmod 4,\ p+q=n, \\ I^+(\frac{1}{2}), & \text{if } p-q \equiv 2 \pmod 4,\ p+q=n. \end{cases}$$

4. The structure of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$

In this section we shall describe the images of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ under the embeddings

$$\begin{aligned} &\Omega^{p,q} \overset{\varphi}{\hookrightarrow} I^+\left(\tfrac{p+q}{2} - \rho_n\right), && p-q \equiv 0 \pmod 4, \\ &\Omega^{p,q}(1) \overset{\psi}{\hookrightarrow} I^+\left(\tfrac{1}{2}\right), && p-q \equiv 0 \pmod 4,\ p+q=n \text{ is even}, \end{aligned}$$

given in Corollary 3.2. The other embeddings given in Corollary 3.2 can be studied in a similar way, and we leave them to the readers. Using the results in [L2], we are able to deduce the module structure of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$. Since all these representations are K-multiplicity free, the images of $\Omega^{p,q}$ and $\Omega^{p,q}(1)$ are given by the direct sums of K-types whose highest weights are those occurring in $\Omega^{p,q}$ and $\Omega^{p,q}(1)$, respectively. Hence to prove our results in this section we only need to verify carefully that the spaces involved contain exactly the same K-types. Since such arguments are elementary and are similar to those used in [LZ], we shall omit all the proofs.

We shall first consider $\Omega^{p,q}$. Let m be a fixed positive even integer. By Corollary 3.2, for every pair (p,q) of nonnegative integers such that $p+q=m$ and $p-q \equiv 0 \pmod 4$, we have

$$\Omega^{p,q} \overset{\varphi}{\hookrightarrow} I^+(\sigma)$$

where $\sigma = \frac{m}{2} - \rho_n$. We need to consider two cases, according to whether n is even or odd.

CASE: *n even.* We first assume that $m \equiv 2 \pmod 4$. Let $k = \frac{n}{2}$. Then $\rho_n = \frac{n+1}{2} = k + \frac{1}{2}$, and $\sigma = \frac{m}{2} - k - \frac{1}{2} = k - \frac{1}{2} - 2k + \frac{m}{2}$, so that $[\sigma] \equiv k \pmod 2$. Let $\Lambda_e^+ = \{\lambda = (\lambda_1, \dots, \lambda_n)\colon\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \text{ is even } \forall j\}$. For $0 \leq a, b \leq k$, let L_{ab} be the set of $\lambda \in \Lambda_e^+$ which satisfy

$$\begin{aligned} &\lambda_{2r-1} \geq -(\alpha + n - 2r + 1), && 1 \leq r \leq a-1, \\ &\lambda_{2r-1} < -(\alpha + n - 2r + 1), && a \leq r \leq k, \\ &\lambda_{2r} > \alpha + 2r - 1, && 1 \leq r \leq b-1, \\ &\lambda_{2r} \leq \alpha + 2r - 1, && b \leq r \leq k, \end{aligned}$$

where $\alpha = -\frac{m}{2}$. If L_{ab} is non-empty then the direct sum of all K-types whose highest weights lie in L_{ab} forms an irreducible constituent of $I^+(\sigma)$. For convenience, we also denote this subspace by L_{ab}. Note that if $L_{1,k+1} \neq \emptyset$, then it is the only finite dimensional irreducible constituent.

PROPOSITION 4.1 (cf. Theorem 5.2 of [L2]): *Let $n = 2k$ be a positive even integer, and let $\sigma \in \frac{1}{2} + \mathbf{Z}$ be such that $[\sigma] \equiv k \pmod 2$.*

(i) *If $\sigma \leq -\frac{1}{2}$, then*

$$I^+(\sigma) = \bigoplus\{L_{ab}\colon\ 0 \leq a - b \leq r_1\},$$

where $r_1 = \min([|\sigma|] + 1, k)$, and the socle series (see page 31) of $I^+(\sigma)$ is given by

$$\mathrm{Soc}^{\,j}(I^+(\sigma)) = \begin{cases} \bigoplus\{L_{ab}\colon\ r_1 - j + 1 \leq a - b \leq r_1\} & 1 \leq j \leq r_1, \\ I^+(\sigma) & j \geq r_1 + 1. \end{cases}$$

Moreover, a constituent L_{ab} of $I^+(\sigma)$ is unitarizable if and only if $a = b$ or $-k + \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$ and $a - b = r_1$.

(ii) *If $\sigma \geq \frac{1}{2}$, then*

$$I^+(\sigma) = \bigoplus\{L_{ab}\colon\ -1 \leq b - a \leq r_2\},$$

where $r_2 = \min([\sigma], k)$, and the socle series of $I^+(\sigma)$ is given by

$$\mathrm{Soc}^{\,j}(I^+(\sigma)) = \begin{cases} \bigoplus\{L_{ab}\colon\ -1 \leq b - a \leq j - 2\} & 1 \leq j \leq r_2 + 1, \\ I^+(\sigma) & j \geq r_2 + 2. \end{cases}$$

Moreover, a constituent L_{ab} of $I^+(\sigma)$ is unitarizable if and only if $a = b + 1$ or $\frac{1}{2} \leq \sigma \leq k + \frac{1}{2}$ and $b - a = r_2$.

SUBCASE: $m \equiv 2 \pmod 4$, $2 \leq m \leq n$. In this case, we have $[\sigma] \equiv k \pmod 2$ and $-k + \frac{1}{2} \leq \sigma \leq -\frac{1}{2}$. Each of the irreducible constituent L_{ab} such that $a - b = -\frac{m}{2} + k + 1$ forms an irreducible unitary submodule of $I^+(\sigma)$ and this accounts for all the irreducible submodules of $I^+(\sigma)$. It turns out that each of these submodules can be identified with some Howe quotient $\Omega^{p,q}$. Note that if p and q are nonnegative integers such that $p + q = m$, then $p - q \equiv 0 \pmod 4$ if and only if both p and q are odd.

THEOREM 4.2: *Assume that n is even. Let $2 \leq m \leq n$ and $m \equiv 2 \pmod 4$. If p and q are positive odd integers such that $p + q = m$, then*

$$\varphi(\Omega^{p,q}) = L_{\frac{n-q+3}{2}, \frac{p+1}{2}} \subseteq I^+\left(\frac{m}{2} - \rho_n\right).$$

Hence $\varphi(\Omega^{p,q})$ is an irreducible unitary submodule of $I^+\left(\frac{m}{2} - \rho_n\right)$. Moreover, the set $\{\varphi(\Omega^{p,q})\colon\ p + q = m,\ p \text{ and } q \text{ are odd and positive}\}$ exhausts all the irreducible unitary submodules of $I^+\left(\frac{m}{2} - \rho_n\right)$.

The module diagram (see [A] or §7 of [L1]) of $I^+(\sigma)$ for $-k+\frac{1}{2} \le \sigma \le -\frac{1}{2}$ is given in Figure 1 below. Thus we see that the Howe quotients $\Omega^{p,q}$ in Theorem 4.2 are mapped to the lowest row of the module diagram.

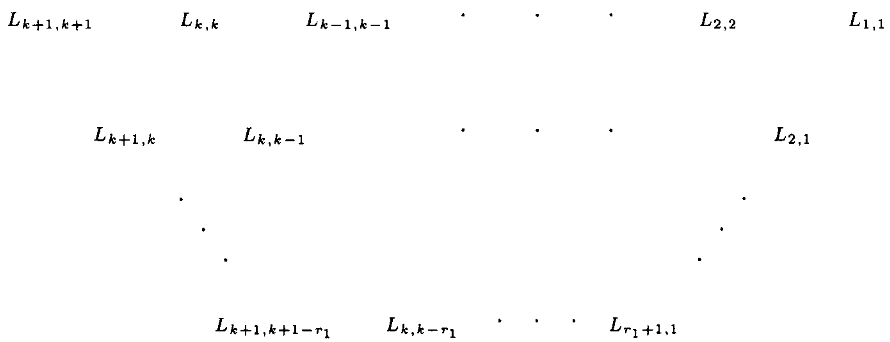

Figure 1.

We consider an example. Let $n=8$ and $m=6$. Then $(5,1)$, $(3,3)$ and $(5,1)$ are all the pairs (p,q) of odd positive integers such that $p+q=6$. For these pairs (p,q), we have

$$\Omega^{p,q} \overset{\varphi}{\hookrightarrow} I^+\left(\frac{6}{2}-\frac{9}{2}\right) = I^+\left(-\frac{3}{2}\right).$$

By Theorem 4.2, we have $\varphi(\Omega^{5,1}) = L_{5,3}$, $\varphi(\Omega^{3,3}) = L_{4,2}$, $\varphi(\Omega^{1,5}) = L_{3,1}$ (see Figure 2 below).

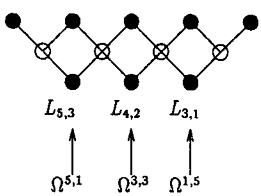

Figure 2.

Here and after, a black circle represents a unitary constituent.

SUBCASE: $m \equiv 2 \pmod 4$, $m \ge n+2$. In this case, $\sigma \ge \frac{n+2}{2} - \frac{n+1}{2} = \frac{1}{2}$. If $\sigma \ge k+\frac{1}{2}$, then the module diagram of $I^+(\sigma)$ is given in Figure 3. If $\frac{1}{2} \le \sigma \le k-\frac{1}{2}$, then the diagram of $I^+(\sigma)$ is the configuration obtained by deleting the top $k-[\sigma]$ rows from Figure 3.

If $L_{s,t}$ is a constituent of $I^+(\sigma)$, we let $M(s,t)$ be the submodule of $I^+(\sigma)$ generated by $L_{s,t}$ (see Figure 4 below). Specifically,

$$M(s,t) = \bigoplus \{L_{a,b}\colon\ a \geq s,\ b \leq t\}.$$

In particular, if $L_{1,k+1} \neq \emptyset$, then $M(1,k+1) = I^+(\sigma)$.

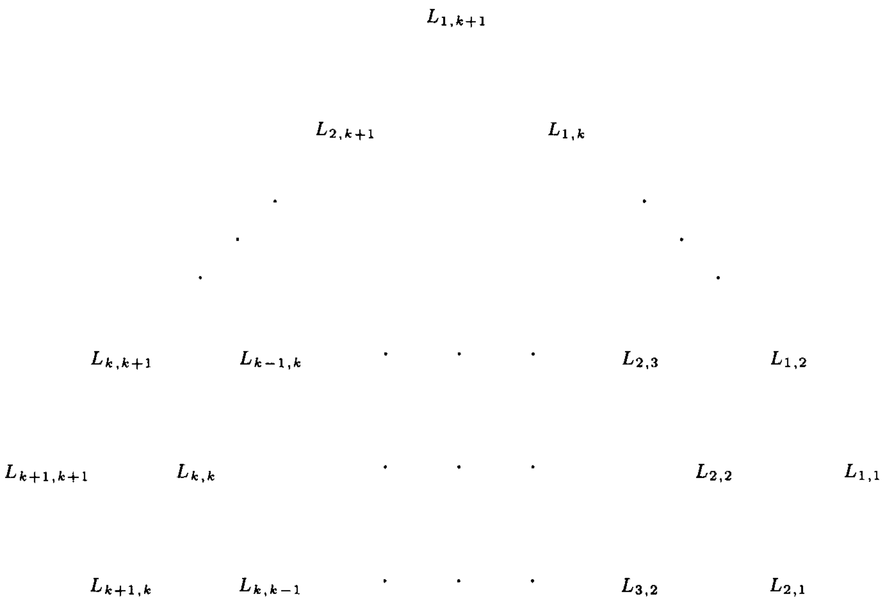

Figure 3.

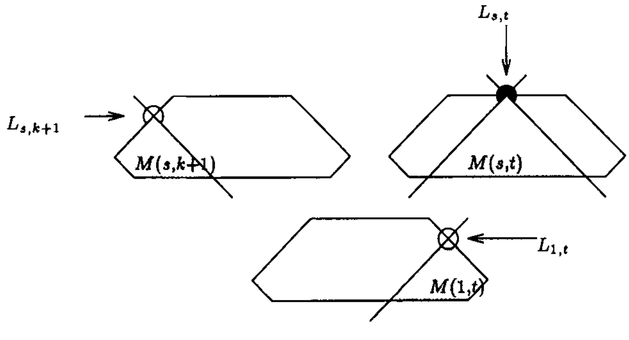

Figure 4.

THEOREM 4.3: *Assume that $n = 2k$ is even. Let m be an even integer such that $m \geq n+2$ and $m \equiv 2 \pmod 4$. If p and q are odd integers such that $p+q=m$, then*

$$\varphi\left(\Omega^{p,q}\right) = M(s,t) \subsetneq I^+\left(\frac{m}{2} - \rho_n\right)$$

and

$$Q^{p,q} \cong L_{s,t},$$

where

$$s = \max\left(\frac{n-q+3}{2}, 1\right), \quad t = \min\left(\frac{p+1}{2}, k+1\right).$$

In particular, we have:

(i) *$\varphi(\Omega^{p,q}) = I^+(\sigma)$ if and only if $Q^{p,q}$ is finite dimensional if and only if $p \geq n+1$ and $q \geq n+1$.*

(ii) *$\Omega^{p,q}$ is reducible.*

(iii) *Q^{pq} is unitary if and only if $p, q \leq n+1$.*

We illustrate this theorem by the following example. Let $n = 8$ and $m = 14$. There are 7 pairs of odd positive integers (p,q) such that $p+q=14$; namely $(13,1)$, $(11,3)$, $(9,5)$, $(7,7)$, $(5,9)$, $(3,11)$ and $(1,13)$. For such pairs (p,q), we have

$$\Omega^{p,q} \overset{\varphi}{\hookrightarrow} I^+\left(\frac{14}{2} - \frac{9}{2}\right) = I^+\left(\frac{5}{2}\right).$$

By Theorem 4.3, we have $\varphi\left(\Omega^{13,1}\right) = M(5,5)$, $\varphi\left(\Omega^{11,3}\right) = M(4,5)$, $\varphi\left(\Omega^{9,5}\right) = M(3,5)$, $\varphi\left(\Omega^{7,7}\right) = M(2,4)$, $\varphi\left(\Omega^{5,9}\right) = M(1,3)$, $\varphi\left(\Omega^{3,11}\right) = M(1,2)$, and $\varphi\left(\Omega^{1,13}\right) = M(1,1)$. We indicate the images of $\Omega^{p,q}$ in the module diagram of

$I^+(5/2)$ in Figure 5.

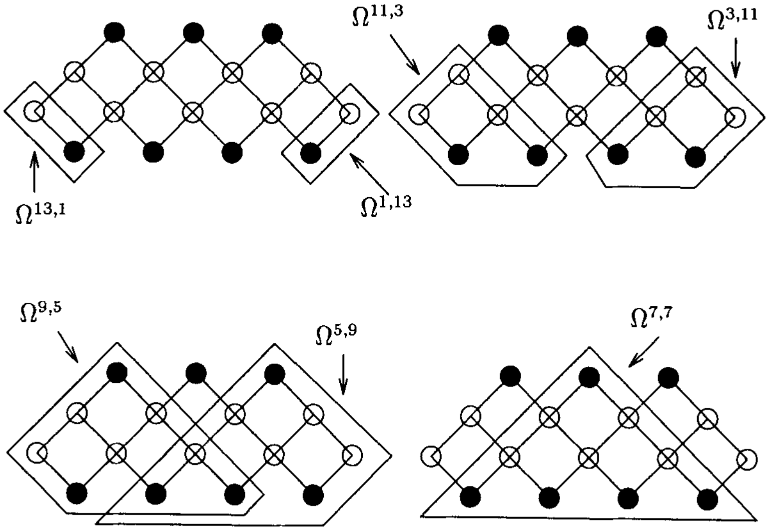

Figure 5.

Next we consider the case when $m \equiv 0 \pmod 4$. In this case $\sigma = \frac{m}{2} - k - \frac{1}{2} = k + \frac{3}{2} + \frac{m}{2} - 2k - 2$ so that $[\sigma] \equiv k + 1 \pmod 2$. As pointed out in [L2], the structure of $I^+(\sigma)$ in this case can be deduced from that of $I^+(-\sigma)$, which is contragradient to $I^+(\sigma)$. Since $[-\sigma] \equiv k \pmod 2$, the structure of $I^+(-\sigma)$ is given in Proposition 4.1. We shall now describe the structure of $I^+(\sigma)$. Suppose L_{ij} is a constituent of $I^+(-\sigma)$. We let

$$L_{ij}^* = \{\lambda \in \Lambda_e^+ \colon \ \lambda^* \in L_{ij}\}.$$

Here if $\lambda = (\lambda_1, ..., \lambda_n)$, then $\lambda^* = (-\lambda_n, -\lambda_{n-1}, ..., -\lambda_1)$. Explicitly $\lambda \in L_{ij}^*$ if and only if

$$\begin{aligned} \lambda_{2r-1} &> \alpha + 2r - 2, && 1 \le r \le k - j + 1, \\ \lambda_{2r-1} &\le \alpha + 2r - 2, && k - j + 2 \le r \le k, \\ \lambda_{2r} &\ge -(\alpha + n - 2r), && 1 \le r \le k - i + 1, \\ \lambda_{2r} &\le -(\alpha + n - 2r), && k - i + 2 \le r \le k, \end{aligned}$$

where $\alpha = -\frac{m}{2}$. As before we identify L_{ij}^* with the subspace of $I^+(\sigma)$ which is the direct sum of all the K-types whose highest weights are contained in L_{ij}^*. Then L_{ij}^* is an irreducible constituent of $I^+(\sigma)$ and $I^+(\sigma) = \oplus\{L_{ij}^* \colon \ L_{ij} \neq \emptyset\}$. We state the following proposition for the convenience of the readers.

Proposition 4.4: *Let $n = 2k$ be an even integer, and let $\sigma \in \frac{1}{2} + \mathbf{Z}$ be such that* $[\sigma] \equiv k + 1 \pmod 2$.

(i) *If $\sigma \geq \frac{1}{2}$, then*

$$I^+(\sigma) = \bigoplus\{L^*_{ab}\colon\ 0 \leq a-b \leq r_1\}$$

where $r_1 = \min([\sigma]+1, k)$, and the socle series of $I^+(\sigma)$ is given by

$$\mathrm{Soc}^{\,j}\left(I^+(\sigma)\right) = \begin{cases} \bigoplus\{L^*_{ab}\colon\ 0 \leq a-b \leq k-1\}, & 1 \leq j \leq r_1, \\ I^+(\sigma), & j \geq r_1+1. \end{cases}$$

(ii) *If $\sigma \leq -\frac{1}{2}$, then*

$$I^+(\sigma) = \bigoplus\{L^*_{ab}\colon\ -1 \leq b-a \leq r_2\}$$

where $r_2 = \min([|\sigma|], k)$, and the socle series of $I^+(\sigma)$ is given by

$$\mathrm{Soc}^{\,j}\left(I^+(\sigma)\right) = \begin{cases} \bigoplus\{L^*_{ab}\colon\ r_2-j+1 \leq b-a \leq r_2\}, & 1 \leq j \leq r_2+1, \\ I^+(\sigma), & j \geq r_2+2. \end{cases}$$

Note that if $m \equiv 0 \pmod 4$ and $p+q=m$, then $p-q \equiv 0 \pmod 4$ if and only if both p and q are even.

SUBCASE: $m \equiv 0 \pmod 4$, $4 \leq m \leq n$. Note that in this case $\sigma \leq -\frac{1}{2}$.

THEOREM 4.5: *Assume that n is even. Let m be an integer such that $4 \leq m \leq n$ and $m \equiv 0 \pmod 4$. Then for any positive even integers p and q such that $p+q=m$, we have*

$$\varphi(\Omega^{p,q}) = L^*_{\frac{q+2}{2},\frac{n-p+2}{2}} \subseteq I^+\left(\frac{m}{2}-\rho_n\right).$$

Hence each $\varphi(\Omega^{p,q})$ is irreducible and unitary. Moreover, the set $\{\varphi(\Omega^{p,q})\colon\ p+q=m,\ p$ and q are even and nonnegative$\}$ exhausts all the irreducible unitary submodules o $I^+(\frac{m}{2}-\rho_n)$.

SUBCASE: $m \equiv 0 \pmod 4$, $m \geq n+2$. In this case $\sigma \geq \frac{1}{2}$. Let L^*_{ij} be an irreducible constituent of $I^+(\sigma)$. Let $M'(i,j)$ be the submodule of $I^+(\sigma)$ generated by L^*_{ij}. Then

$$M'(i,j) = \oplus\{L^*_{ab}\colon\ a \leq i,\ b \geq j\}.$$

THEOREM 4.6: *Assume that $n = 2k$ is even. Let m be an integer such that $m \geq n+2$ and $m \equiv 0 \pmod 4$. Then for any pair of positive even integers p and q such that $p+q=m$, we have*

$$\varphi\left(\Omega^{p,q}\right) = M'(s,t),$$

and

$$Q^{p,q} \cong L^*_{s,t},$$

where

$$s = \min\left(\frac{q+2}{2}, k+1\right), \text{ and } t = \max\left(\frac{n-p+2}{2}, 1\right).$$

In particular, we have

(i) $\varphi(\Omega^{p,q}) = I^+(\sigma)$ *if and only if* $p \geq n$ *and* $q \geq n$.

(ii) $\Omega^{p,q}$ *is irreducible if and only if* $pq = 0$.

(iii) $Q^{p,q}$ *is not finite dimensional.*

(iv) $Q^{p,q}$ *is unitary if and only if either* $pq = 0$ *or* $p, q \leq n$.

CASE: n *odd.* Let $n = 2k+1$. Again we shall first consider the case when $m \equiv 2 \pmod 4$. Note that in this case we have $\sigma = \frac{m}{2} - k - 1 \equiv k \pmod 2$.

SUBCASE: $2 \leq m \leq n-1$, $m \equiv 2 \pmod 4$. In this case we have $-k \leq \sigma \leq -1$. For nonnegative integers s and t such that $s + t \leq k$, we let $R_{a(s,t)}$ be the set of $\lambda \in \Lambda_e^+$ which satisfy

$$\begin{array}{ll} \lambda_{2r} > \alpha + 2r - 1, & 1 \leq r \leq s, \\ -(\alpha + n - 2r) \leq \lambda_{2r} \leq \alpha + 2r - 1, & s+1 \leq r \leq k-t, \\ \lambda_{2r} < -(\alpha + n - 2r), & k-t+1 \leq r \leq k, \end{array}$$

where $\alpha = -m/2$. If $R_{a(s,t)} \neq \emptyset$, then the direct sum of all K-types whose highest weights lie in $R_{a(s,t)}$ form an irreducible constituent of $I^+(\sigma)$ and we shall denote this constituent also by $R_{a(s,t)}$. Notice that $R_{a(s,t)}$ is not finite dimensional.

PROPOSITION 4.7 (cf. Theorem 5.4 of [L2]): *Let* $n = 2k+1$ *be an odd integer and let* σ *be a negative integer such that* $\sigma \equiv k \pmod 2$. *Then*

$$I^+(\sigma) = \bigoplus\{R_{a(s,t)}\colon\ k - r \leq s + t \leq k\},$$

where $r = \min(|\sigma|, k)$, *and the socle series of* $I^+(\sigma)$ *is given by*

$$\operatorname{Soc}^j(I^+(\sigma)) = \begin{cases} \bigoplus\{R_{a(s,t)}\colon k - r \leq s + t \leq k - r + j - 1\} & 1 \leq j \leq r, \\ I^+(\sigma) & j \geq r+1. \end{cases}$$

Moreover, we have:

(i) *For* s *and* t *such that* $s + t = k - r$, $R_{a(s,t)}$ *is an irreducible submodule of* $I^+(\sigma)$;

(ii) *A constituent $R_{a(s,t)}$ is unitarizable if and only if $-k \le \sigma \le -1$ and $s+t = k+\sigma$.*

THEOREM 4.8: *Assume that n is odd. Let m be an integer such that $2 \le m \le n-1$ and $m \equiv 2 \pmod 4$. If p and q are odd integers such that $p+q=m$, then*

$$\varphi\left(\Omega^{p,q}\right) = R_{a\left(\frac{p-1}{2},\frac{q-1}{2}\right)} \subseteq I^+\left(\frac{m}{2} - \rho_n\right).$$

Hence each $\varphi(\Omega^{p,q})$ is an irreducible unitary submodule of $I^+(\frac{m}{2} - \rho_n)$. Moreover, the set $\{\varphi(\Omega^{p,q})$: $p+q=m$, p and q are positive and odd$\}$ exhausts all the irreducible unitary submodules of $I^+\left(\frac{m}{2} - \rho_n\right)$.

We now illustrate this theorem with an example. Let $n=9$ and $m=6$. If (p,q) is one of the pairs $(5,1)$, $(3,3)$ and $(1,5)$, then

$$\Omega^{p,q} \hookrightarrow I^+\left(\frac{6}{2} - \frac{10}{2}\right) = I^+(-2).$$

The module diagram of $I^+(-2)$ is given in Figure 6.

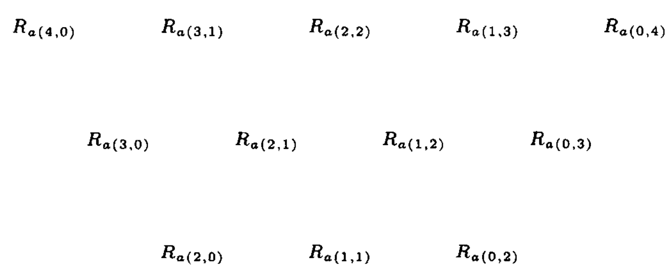

Figure 6.

By Theorem 4.8, we have $\varphi(\Omega^{5,1}) = R_{a(2,0)}$, $\varphi(\Omega^{3,3}) = R_{a(1,1)}$, and $\varphi(\Omega^{1,5}) = R_{a(0,2)}$. Thus $\Omega^{5,1}$, $\Omega^{3,3}$ and $\Omega^{1,5}$ are mapped to the lowest row in the module diagram of $I^+(-2)$.

SUBCASE: $m = n+1$, $m \equiv 2 \pmod 4$. In this case, $n \equiv 1 \pmod 4$ and $\sigma = 0$. Note that $I^+(0)$ is on the unitary axis. In fact, by Theorem 5.5 of [L2], we have

$$I^+(0) = \bigoplus_{j=0}^{k} U_j,$$

where for each $0 \leq j \leq k$, U_j is the direct sum of all K-types whose highest weights lie in the set

$$\{\lambda \in \Lambda_e^+ \colon \ \lambda_{2j+2} \leq -k + 2j \leq \lambda_{2j}\}.$$

Each U_j is an irreducible unitary submodule of $I^+(0)$.

THEOREM 4.9: *If $n \equiv 1 \pmod 4$ and if p and q are odd integers such that $p + q = n + 1$, then*

$$\varphi(\Omega^{p,q}) = U_{\frac{p-1}{2}}.$$

In particular,

$$I^+(0) = \bigoplus \{\varphi(\Omega^{p,n+1-p}) \colon \ 1 \leq p \leq n, p \ odd\}.$$

SUBCASE: $m \equiv 2 \pmod 4$, $m \geq n + 3$. In this case, we have $\sigma \geq 1$. As pointed out in [L2], the structure of $I^+(\sigma)$ can be deduced from the dual module $I^+(-\sigma)$. We shall give now a brief summary on the structure of $I^+(\sigma)$ here. Suppose that $R_{a(s,t)}$ be an irreducible constituent of $I^+(-\sigma)$. Let

$$R^*_{a(s,t)} = \{\lambda \in \Lambda_e^+ \colon \ \lambda^* \in R_{a(s,t)}\}.$$

Explicitly, $\lambda \in R^*_{a(s,t)}$ if and only if

$$\begin{array}{ll} \lambda_{2r} \geq -(\alpha + n - 2r), & 1 \leq r \leq t, \\ \alpha + 2r - 1 \leq \lambda_{2r} \leq -(\alpha + n - 2r), & t + 1 \leq r \leq k - s, \\ \lambda_{2r} < \alpha + 2r - 1, & k - s + 1 \leq r \leq k, \end{array}$$

where $\alpha = -m/2$. If $\sigma \geq k$, the module diagram of $I^+(\sigma)$ is a triangle, as given in Figure 7.

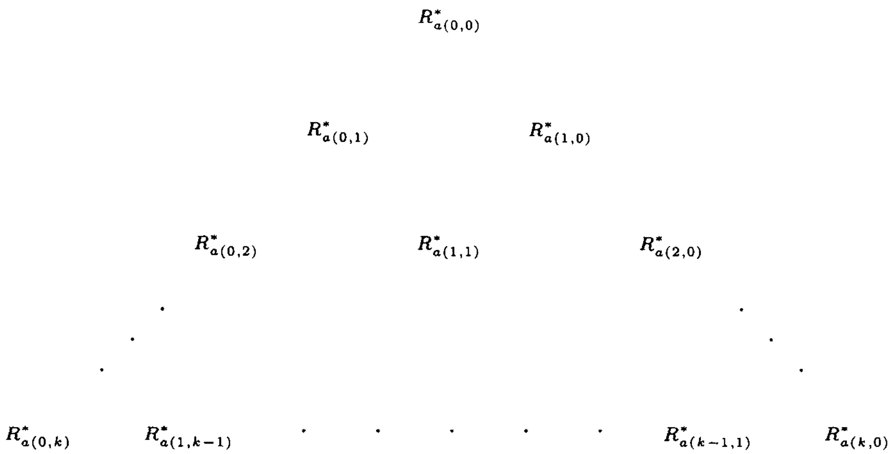

Figure 7.

If $1 \leq \sigma \leq k-1$, then the module diagram of $I^+(\sigma)$ is the triangle with its top $k-\sigma$ rows deleted. For any integral $\sigma \geq 1$, each of the irreducible constituents at the lowest row forms an irreducible unitary submodule of $I^+(\sigma)$. If $1 \leq \sigma \leq k$, then the irreducible constituents at the top rows are also unitary.

If $R^*_{a(s,t)}$ is an irreducible constituent of $I^+(\sigma)$, we let $S(s,t)$ be the submodule of $I^+(\sigma)$ generated by $R_{a(s,t)}$. Specifically, we have

$$S(s,t) = \bigoplus \left\{ R^*_{a(j,k)} \colon\ j \geq s,\ k \geq t \right\}.$$

The module diagram of $S(s,t)$ is a subtriangle in the module diagram of $I^+(\sigma)$. Note that $S(0,0) = I^+(\sigma)$.

THEOREM 4.10: *Assume that n is odd. Let m be an integer such that $m \geq n+3$ and $m \equiv 2 \pmod 4$. If p and q are positive odd integers such that $p+q=m$, then we have*

$$\varphi\left(\Omega^{p,q}\right) = S(s,t) \subseteq I^+\left(\frac{m}{2} - \rho_n\right),$$

and

$$Q^{p,q} \cong R^*_{a(s,t)},$$

where $s = \max\left(0, \frac{n-p}{2}\right)$ and $t = \max\left(0, \frac{n-q}{2}\right)$. In particular, we have:

(i) *$\varphi(\Omega^{p,q}) = I^+(\sigma)$ if and only if $p \geq n$ and $q \geq n$.*

(ii) *$\Omega^{p,q}$ is irreducible if and only if either $n=1$ or $n>1$ and $(p-1)(q-1)=0$.*

(iii) *$Q^{p,q}$ is not finite dimensional.*

(iv) *$Q^{p,q}$ is unitary if and only if $p, q \leq n$.*

We now illustrate Theorem 4.10 with an example. Let $n = 9$ and $m = 14$. If p and q are odd integers such that $p + q = 14$, then

$$\Omega^{p,q} \hookrightarrow I^+\left(\frac{14}{2} - \frac{10}{2}\right) = I^+(2).$$

Now by Theorem 4.10, we have $\varphi\left(\Omega^{13,1}\right) = S(0,4)$, $\varphi\left(\Omega^{11,3}\right) = S(0,3)$, $\varphi\left(\Omega^{9,5}\right) = S(0,2)$, $\varphi\left(\Omega^{7,7}\right) = S(1,1)$, $\varphi\left(\Omega^{5,9}\right) = S(2,0)$, $\varphi\left(\Omega^{3,11}\right) = S(3,0)$, and $\varphi\left(\Omega^{1,13}\right) = S(4,0)$. The images of these $\Omega^{p,q}$ in the module diagram of $I^+(2)$ are given in Figure 8.

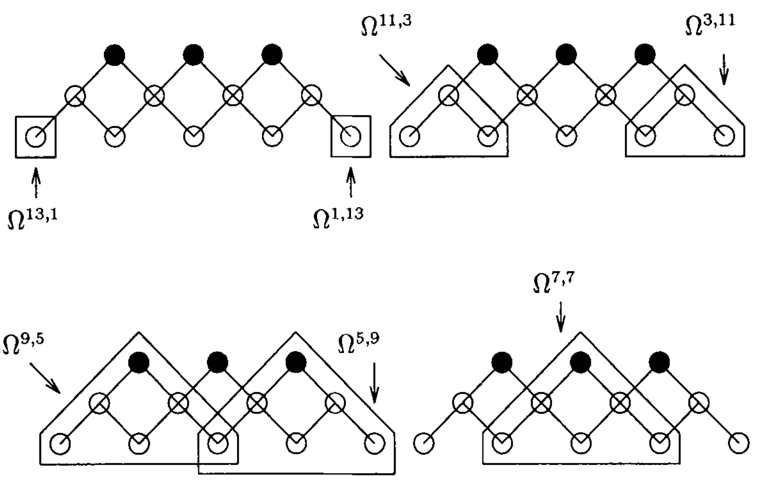

Figure 8.

Next we consider the case when $m \equiv 0 \pmod 4$. In this case $\sigma = \frac{m}{2} - k - 1 \equiv k + 1 \pmod 2$.

SUBCASE: $m \equiv 0 \pmod 4$, $4 \leq m \leq n - 1$. Under these assumptions, $\sigma \leq -1$. We recall the structure of $I^+(\sigma)$ in this case. For nonnegative integers s and t such that $s + t \leq k + 1$, let $W_{a(s,t)}$ be the set of all $\lambda \in \Lambda_e^+$ which satisfy the following conditions:

$$\begin{aligned} &\lambda_{2r-1} > \alpha + 2r - 2, && 1 \leq r \leq s, \\ &-(\alpha + n - 2r + 1) \leq \lambda_{2r-1} \leq \alpha + 2r - 2, && s + 1 \leq r \leq k - t + 1, \\ &\lambda_{2r-1} < -(\alpha + n - 2r + 1), && k - t + 2 \leq r \leq k + 1, \end{aligned}$$

where $\alpha = -\frac{m}{2}$. As usual, if $W_{a(s,t)} \neq \emptyset$, we identify $W_{a(s,t)}$ with the subspace of $I^+(\sigma)$ which is the direct sum of all K-types whose highest weights lie in $W_{a(s,t)}$. Note that among the nonempty $W_{a(s,t)}$'s, only $W_{a(0,0)}$ is finite dimensional.

THEOREM 4.11: *Assume that n is odd. Let m be an integer such that $4 \leq m \leq n-1$ and $m \equiv 0 \pmod 4$. If p and q are nonnegative even integers such that $p+q=m$, then*

$$\varphi\left(\Omega^{p,q}\right) = W_{a(\frac{p}{2},\frac{q}{2})} \subseteq I^+\left(\frac{m}{2} - \rho_n\right).$$

Hence each $\varphi(\Omega^{p,q})$ is an irreducible unitary submodule of $I^+(\frac{m}{2}-\rho_n)$. Moreover the set $\{\varphi(\Omega^{p,q})\colon\ p+q=m,\ \ p$ and q are nonnegative and even $\}$ exhausts all the irreducible unitary submodules of $I^+(\frac{m}{2}-\rho_n)$.

SUBCASE: $m \equiv 0 \pmod 4$, $m = n+1$. In this case, $n \equiv 3 \pmod 4$ and $\sigma = 0$. $I^+(0)$ is on the unitary axis and

$$I^+(0) = \bigoplus_{j=0}^{k+1} F_j,$$

where for each $0 \leq j \leq k+1$, F_j is the direct sum of all K-types whose highest weights lie in the set

$$\{\lambda \in \Lambda_e^+\colon\ \lambda_{2j+1} \leq -k+2j-1 \leq \lambda_{2j-1}\}.$$

Each F_j is an irreducible unitary submodule of $I^+(0)$.

THEOREM 4.12: *If $n \equiv 3 \pmod 4$ and if p and q are nonnegative even integers such that $p+q=n+1$, then we have*

$$\varphi(\Omega^{p,q}) = F_{\frac{p}{2}} \subseteq I^+(0).$$

In particular,

$$I^+(0) = \bigoplus\{\varphi(\Omega^{p,n+1-p})\colon\ 0 \leq p \leq n+1, p \text{ even}\}.$$

SUBCASE: $m \geq n+3$, $m \equiv 0 \pmod 4$. In this case, $\sigma \geq 1$. Suppose $W_{a(s,t)}$ is a constituent in $I^+(-\sigma)$. Let

$$W^*_{a(s,t)} = \{\lambda \in \Lambda_e^+\colon\ \lambda^* \in W_{a(s,t)}\}.$$

Explicitly, $\lambda \in W^*_{a(s,t)}$ if and only if

$$\begin{array}{ll} \lambda_{2r-1} \geq -(\alpha+n-2r+1), & 1 \leq r \leq t, \\ \alpha+2r-2 < \lambda_{2r-1} < -(\alpha+n-2r+1), & t+1 \leq r \leq k+1-s, \\ \lambda_{2r-1} \leq \alpha+2r-2, & k+2-s \leq r \leq k+1, \end{array}$$

where $\alpha = -\frac{m}{2}$. If $\sigma \geq k+1$, the module diagram of $I^+(\sigma)$ is a triangle, as given in Figure 9.

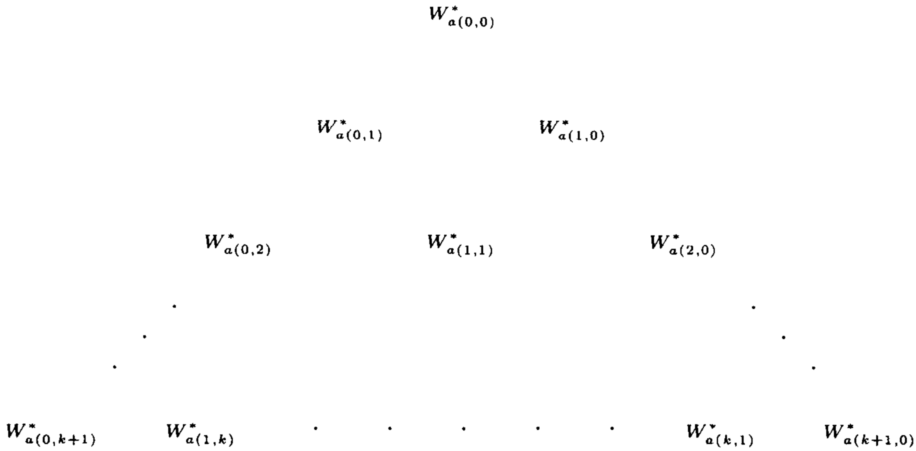

Figure 9.

If $1 \leq \sigma \leq k$, then the module diagram of $I^+(\sigma)$ is the triangle with its top $k+1-\sigma$ rows deleted. For any integral $\sigma \geq 1$, each of the irreducible constituents at the lowest row forms an irreducible submodule of $I^+(\sigma)$, and only $W^*_{a(0,k+1)}$ and $W^*_{a(k+1,0)}$ are unitary. If $1 \leq \sigma \leq k+1$, then the irreducible constituents at the top rows are also unitary.

If $W^*_{a(s,t)}$ is an irreducible constituent of $I^+(\sigma)$, we let $X(s,t)$ be the submodule of $I^+(\sigma)$ generated by $W^*_{a(s,t)}$. Specifically, we have

$$X(s,t) = \bigoplus \left\{ W^*_{a(i,j)}\colon\ i \geq s,\ j \geq t \right\}.$$

The module diagram of $X(s,t)$ is a subtriangle in the module diagram of $I^+(\sigma)$. Note that $W^*_{a(0,0)}$ is finite dimensional and $X(0,0) = I^+(\sigma)$.

THEOREM 4.13: *Assume that n is odd. Let m be an integer such that $m \geq n+3$ and $m \equiv 0 \pmod 4$. If p and q are nonnegative even integers such that $p+q=m$, then we have*

$$\varphi(\Omega^{p,q}) = X(s,t),$$

and

$$Q^{p,q} \cong W^*_{a(s,t)},$$

where $s = \max\left(0, \frac{n+1-p}{2}\right)$ and $t = \max\left(0, \frac{n+1-q}{2}\right)$. In particular, we have:

(i) *$\varphi(\Omega^{p,q}) = I^+(\sigma)$ if and only if $Q^{p,q}$ is finite dimensional if and only if $p \geq n+1$ and $q \geq n+1$.*

(ii) *$\Omega^{p,q}$ is irreducible if and only if $pq = 0$.*

(iii) *$Q^{p,q}$ is unitary if and only if either $pq = 0$ or $p, q \leq n+1$.*

Finally we consider $\Omega^{p,q}(1)$. By Corollary 3.2, if n is even, and p and q are nonnegative even integers such that $p + q = n$, then

$$\Omega^{p,q}(1) \overset{\psi}{\hookrightarrow} I^{+}\left(\frac{1}{2}\right).$$

We first assume that $n \equiv 0 (\mathrm{mod}\ 4)$. Then the module diagram for $I^{+}(\frac{1}{2})$ is given in Figure 10.

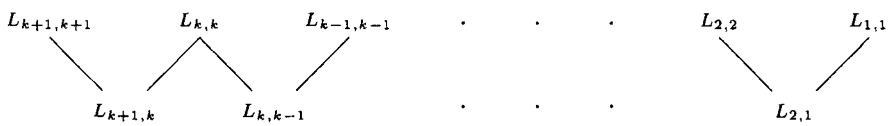

Figure 10.

THEOREM 4.14: *Let $n \equiv 0$ (mod 4). If p is an odd integer such that $1 \leq p \leq n-1$, then*

$$\psi\left(\Omega^{p,n-p}(1)\right) = L_{\frac{p+3}{2},\frac{p+1}{2}} \subseteq I^{+}(\tfrac{1}{2}).$$

The set $\{\psi\left(\Omega^{p,n-p}(1)\right) \colon 1 \leq p \leq n-1, p \text{ odd}\}$ exhausts all the irreducible unitary submodules of $I^{+}(\frac{1}{2})$

Theorems 4.3 and 4.14 reveal some interesting phenomenon. By Theorem 4.3, if p and q are positive odd integers such that $p + q = n + 2$, we also have $\varphi\left(\Omega^{p,q}\right) \hookrightarrow I^{+}(\frac{1}{2})$. More precisely, we have $\varphi\left(\Omega^{n+1,1}\right) = M(k+1, k+1)$ and $\varphi\left(\Omega^{1,n+1}\right) = M(1,1)$. On the other hand, $\psi\left(\Omega^{n-1,1}(1)\right) = L_{k+1,k}$. Hence we conclude that $\Omega^{n-1,1}(1)$ is isomorphic to the unique irreducible submodule of $\Omega^{n+1,1}$, and $\Omega^{1,n-1}(1)$ is isomorphic to the unique irreducible submodule of $\Omega^{1,n+1}$. Similarly, for $3 \leq p \leq n-3$, since $\varphi\left(\Omega^{p,n+2-p}\right) = M\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$, $\Omega^{p,n-p}(1)$ and $\Omega^{p-2,n-p+2}(1)$ are isomorphic to the two irreducible submodules in $\Omega^{p,n+2-p}$.

THEOREM 4.15: *Suppose that $n \equiv 0$ (mod 4).*

(i) *$\Omega^{n-1,1}(1)$ is isomorphic to the unique irreducible submodule of $\Omega^{n+1,1}$.*

(ii) *If p is an odd integer such that $3 \leq p \leq n-3$, then $\Omega^{p,n-p}(1)$ and $\Omega^{p-2,n-p+2}(1)$ are isomorphic to the two irreducible submodules of $\Omega^{p,n+2-p}$, respectively.*

(iii) *$\Omega^{1,n-1}(1)$ is isomorphic to the unique irreducible submodule of $\Omega^{1,n+1}$.*

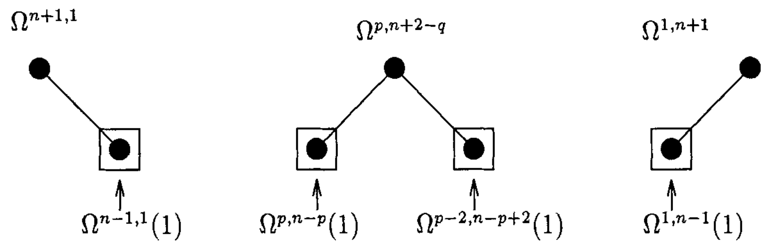

Figure 11.

If $n \equiv 2 \pmod 4$, then the module diagram for $I^+(\frac{1}{2})$ is given in Figure 12.

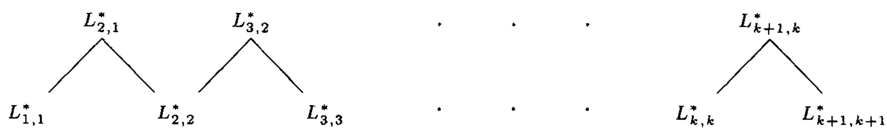

Figure 12.

THEOREM 4.16: *Let $n \equiv 2 \pmod 4$. If p is an even integer such that $0 \leq p \leq n$, then*

$$\psi\left(\Omega^{p,n-p}(1)\right) = L^*_{\frac{n-p}{2}+1,\frac{n-p}{2}+1} \subseteq I^+(\tfrac{1}{2}).$$

The set $\{\psi\left(\Omega^{p,n-p}(1)\right)\colon\ 0 \leq p \leq n\ \ p \text{ even}\}$ exhausts all the irreducible unitary submodules of $I^+(\frac{1}{2})$.

We note by Theorem 4.6 that

$$\varphi(\Omega^{n+2,0}) = M'(1,1) = L^*_{1,1} = \psi\left(\Omega^{n,0}(1)\right),$$

and

$$\varphi(\Omega^{0,n+2}) = M'(k+1,k+1) = L^*_{k+1,k+1} = \psi\left(\Omega^{0,n}(1)\right),$$

and for $2 \leq p \leq n$, p even, we have

$$\varphi\left(\Omega^{p,n+2-p}\right) = M'\left(\frac{n-p}{2}+2, \frac{n-p}{2}+2\right),$$

which has two irreducible submodules isomorphic to $\Omega^{p,n-p}(1)$ and $\Omega^{p-2,n-p+2}(1)$, respectively (see Figure 13).

THEOREM 4.17: *Suppose that $n \equiv 2 \pmod 4$.*

(i) *$\Omega^{n,0}(1)$ is isomorphic to $\Omega^{n+2,0}$.*

(ii) *If p is an even integer such that $2 \leq p \leq n-2$, then $\Omega^{p,n-p}(1)$ and $\Omega^{p-2,n-p+2}(1)$ are isomorphic to the two irreducible submodules of $\Omega^{p,n+2-p}$, respectively.*

(iii) *$\Omega^{0,n}(1)$ is isomorphic to $\Omega^{0,n+2}$.*

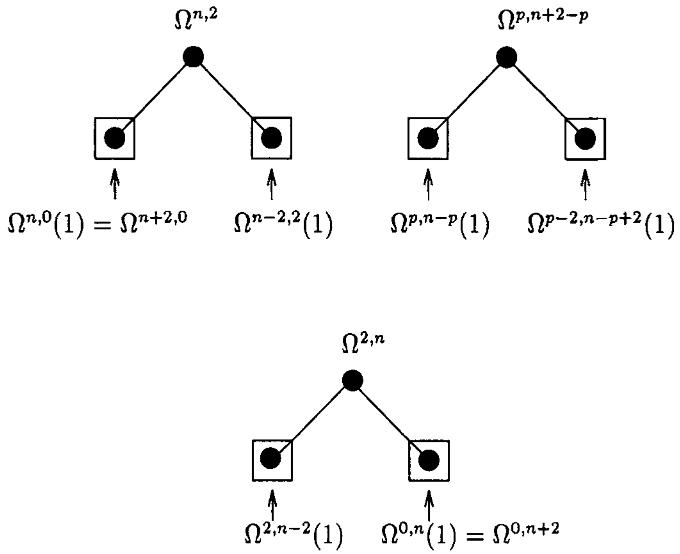

Figure 13.

5. Gelfand–Kirillov dimension of irreducible constituents

Let V be an irreducible constituent of $I^+(\sigma)$ and $V|_K \cong \sum_{\lambda \in R} \rho^\lambda$ where $R \subseteq \Lambda_e^+$. By estimating the sum

$$\sum_{\substack{\lambda \in R \\ |\lambda_1|+\cdots+|\lambda_n|=l}} \dim(\rho^\lambda),$$

as a polynomial in l as in section 6 of [LZ] for the case of $U(n,n)$, we derive the following.

THEOREM 5.1: *Let V be an irreducible constituent of $I^+(\sigma)$ and let $N(V)$ be an integer defined in the following table.*

n	σ	V	$N(V)$
$n = 2k$	$\sigma \in \frac{1}{2} + \mathbf{Z}$	$L_{a,b}$ or $L^*_{a,b}$ *with* $a - b = 0, 1$	n
		$L_{a,b}$ or $L^*_{a,b}$ *with* $a - b \geq 2$	$n - 2(a - b - 1)$
		$L_{a,b}$ or $L^*_{a,b}$ *with* $b - a \geq 1$	$n - 2(b - a)$
$n = 2k + 1$	$\sigma \equiv k$ (mod 2)	$R_{a(s,t)}$ or $R^*_{a(s,t)}$ *with* $s + t = k$	n
		$R_{a(s,t)}$ or $R^*_{a(s,t)}$ *with* $0 \leq s + t < k$	$2(s + t + 1)$
		U_j *for* $0 \leq j \leq k$	n
	$\sigma \equiv k + 1$ (mod 2)	$W_{a(s,t)}$ or $W^*_{a(s,t)}$ *with* $s + t = k + 1$	n
		$W_{a(s,t)}$ or $W^*_{a(s,t)}$ *with* $0 \leq s + t \leq k$	$2(s + t)$
		F_j *for* $0 \leq j \leq k + 1$	n

Then the Gelfand-Kirillov dimension of V *is equal to* $N(V)(n - \frac{N(V)-1}{2})$.

Thus, for example, if $n = 2k$, $\sigma \in \frac{1}{2} + \mathbf{Z}$, $\sigma \geq \frac{1}{2}$, and $[\sigma] \equiv k \pmod 2$, the irreducible constituents of $I^+(\sigma)$ are L_{ab} with $-1 \leq b - a \leq \min([\sigma], k)$. In the lowest row of the module diagram of $I^+(\sigma)$, where $b - a = -1$, the G-K dimension of each of the k constituents is $\frac{n(n+1)}{2}$, and in the next higher level (where $b - a = 0$), the G-K dimension for the $k + 1$ constituents remains the same. Further up the G-K dimensions of each level will strictly decrease. If $\sigma \geq k + \frac{1}{2}$, then the G-K dimension of the constituents in all the level will take the values $N(n - \frac{N-1}{2})$, where $N = n, n, n-2, n-4, \ldots, 2, 0$.

References

[A] J. Alperin, *Diagrams for modules*, The Journal of Pure and Applied Algebra **16** (1980), 111–119.

[Ba] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, I*, Communications on Pure and Applied Mathematics **14** (1961), 187–214.

[BGG] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Model of representations of Lie groups*, Selecta Mathematica Formerly Sovietica **1** (1981), 121–142.

[Fo] G. Folland, *Harmonic analysis in Phase Space*, Annals of Mathematics Studies **122**, Princeton University Press, Princeton, 1989.

[Ge] S. Gelbart, *Examples of dual pairs*, in *Automorphic Forms, Representations and L-functions*, Proceedings of Symposia in Pure Mathematics, Vol. 33, Part 1, American Mathematical Society, Providence, RI, 1979, pp. 287–296.

[GW] K. Goodearl and R. Warfield, *An Introduction to Noncommutative Rings*, London Mathematical Society Student Texts, Vol. 16, Cambridge University Press, Cambridge, 1989.

[H1] R. Howe, *Remarks on classical invariant theory*, Transactions of the American Mathematical Society **313** (1989), 539–570.

[H2] R. Howe, *Transcending classical invariant theory*, Journal of the American Mathematical Society **2** (1989), 535–552.

[H3] R. Howe, *Dual pairs in physics: Harmonic oscillators, photons, electrons, and singletons*, Lectures in Applied Mathematics, Vol. 21, American Mathematical Society, Providence, RI, 1985, pp. 179–206.

[H4] R. Howe, *θ-series and invariant theory*, in *Automorphic forms, representations and L-functions*, Proceedings of Symposia in Pure Mathematics, Vol. 33, Part 1, American Mathematical Society, Providence, RI, 1979, pp. 275–286.

[H5] R. Howe, *Oscillator representation: algebraic and analytic preliminaries*, preprint

[J1] K. Johnson, *Degenerate principal series and compact groups*, Mathematische Annalen **287** (1990), 703–718.

[J] K. Johnson, *Degenerate principal series on tube type domains*, Colloquium Mathematicum **138** (1992), 175–187.

[K] S. Kudla, *Some extensions of the Siegel–Weil formula*, preprint

[KR1] S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel Journal of Mathematics **69** (1990), 25–45.

[KR2] S. Kudla and S. Rallis, *Ramified degenerate principal series representations for* Sp(n), Israel Journal of Mathematics **78** (1992), 209–256.

[KRS] S. Kudla, S. Rallis and D. Soudry, *On the degree 5 L-function for* Sp(2), Inventiones mathematicae **107** (1992), 483–541.

[Ku] S. Kudla, *Seesaw dual reductive pairs*, Progress in Mathematics **46** (1983), 244–268.

[KV1] M. Kashiwara and M. Vergne, *On the Segal–Shale–Weil representations and harmonic polynomials*, Inventiones mathematicae **44** (1978), 1–47.

[KV2] M. Kashiwara and M. Vergne, *Functions on the Shilov boundary of the generalized half plane*, in *Non-commutative Harmonic Analysis*, Lecture Notes in Mathematics **728**, Springer-Verlag, Berlin, 1979, pp. 136–176.

[L1] S. Lee, *On some degenerate principal series representations of $U(n,n)$*, Journal of Functional Analysis **126** (1994), 305–366.

[L2] S. Lee, *Degenerate principal series representations of $Sp(2n,\mathbf{R})$*, Compositio Mathematica, to appear.

[Li] J. Li, *Singular unitary representations of classical groups*, Inventiones mathematicae **97** (1989), 237–255.

[LZ] S. Lee and C. Zhu, *Degenerate principal series and local theta correspondence*, submitted.

[P] T. Przebinda, *The Oscillator duality correspondence for the pair $O(2,2)$*, $\mathrm{Sp}(2,\mathbf{R})$, Memoirs of the American Mathematical Society **403** (1989), 1–105.

[S] S. Sahi, *Unitary representations on the Shilov boundary of a symmetric tube domain*, in *Representations of Groups and Algebras*, Contemporary Mathematics, Volume 145, American Mathematical Society, Providence, 1993, pp. 275–286.

[V] D. Vogan, *Gelfand–Kirillov Dimension for Harish–Chandra Modules*, Inventiones mathematicae **48** (1978), 75–98.

[Wey] H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, N.J., 1946.

[Wei] A. Weil, *Sur certains groups d'operateurs unitaires*, Acta Scientiarum Mathematicarum (Szeged) **111** (1964), 143–211.

[Zha] G. Zhang, *Jordan algebras and generalized principal series representation*, Mathematische Annalen, to appear.

[Zhe] D. P. Zhelobenko, *Compact Lie Groups and their Representations*, Translations of Mathematical Monographs, Vol. **40**, American Mathematical Society, 1973.

[Zhu] C. Zhu, *Invariant distributions of classical groups*, Duke Mathematical Journal **65** (1992), 85–119.